Semidefinite programming converse bounds for quantum communication

arXiv:1709.00200

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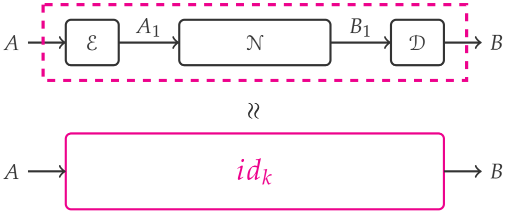
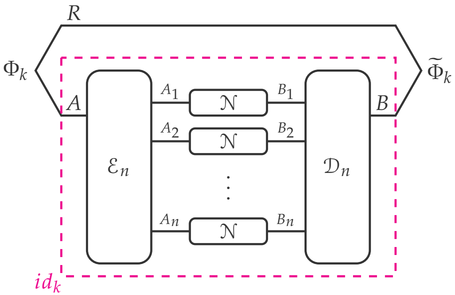
Quantum communication



How well the simulation is? [Kretschmann, Werner, 2004]

- Channel distance $\|\mathcal{D} \circ \mathcal{N} \circ \mathcal{E} - id_k\|_\diamond$.
- Channel fidelity $F\left(\Phi_k, \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}\left(\Phi_k\right)\right)$. ✓,
  where Φ_k is k-dimensional maximally entangled state.
- ...

Quantum capacity



- r: qubits transmitted per channel use.
- n: number of channel copies.
- ε: error tolerance.

- A triplet (r, n, ε) is achievable if $\exists\, \Phi_k, \mathcal{E}_n$ and $\mathcal{D}_n$ such that
$$\frac{1}{n}\log k \geq r, \quad F\left(\Phi_k, \widetilde{\Phi}_k\right) \geq 1-\varepsilon.$$
- Optimal achievable rate given n, ε
$$r^*(n,\varepsilon) := \max\{r : (r,n,\varepsilon) \text{ achievable}\}.$$
- Quantum capacity
$$Q(\mathcal{N}) := \lim_{\varepsilon\to 0}\lim_{n\to\infty} r^*(n,\varepsilon).$$

Theorem (Barnum, Nielsen, Schumacher, 1996-2000; Lloyd, Shor, Devetak, 1997-2005)

For any quantum channel $\mathcal{N}$, it quantum capacity is equal to the regularized coherent information of the channel:

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} I_c\left(\mathcal{N}^{\otimes n}\right),$$

where $I_c(\mathcal{N}) = \max_{\phi_{AA'}} I(A\rangle B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})}$ *and* $\phi_{AA'}$ *pure state.*

- Not a single-letter formula.
- $I_c(\mathcal{N})$ not additive in general.

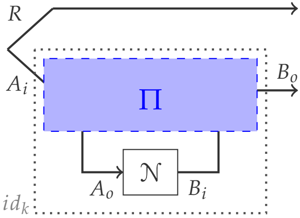
Known converse bounds

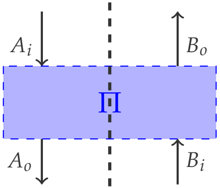
	Strong converse	Efficiently computable	For general channels
R	✓	? (max-min)	✓
ε-DEG	?	✓	✗
E_C	✓	? (regularization)	✓
Q_E	✓	✓	✓
Q_{ss}	?	? (unbounded dimension)	✓
Q_Θ	✓	✓	✓

- R: Rains information [Tomamichel, Wilde, Winter, 2017]
- ε-DEG: Epsilon degradable bound [Sutter, Scholz, Winter, Renner, 2014]
- E_C: Channel's entanglement cost [Berta, Brandão, Christandl,Wehner, 2013]
- Q_E: Entanglement assisted quantum capacity [Bennett, Devetak, Harrow, Shor, Winter,2014; Berta, Christandl, Renner,2011]
- Q_{ss}: Quantum capacity with symmetric side channels [Smith, Smolin, Winter, 2008]
- Q_Θ: Partial transposition bound [Holevo,Werner, 2001]

One-shot quantum capacity

One-shot quantum capacity





$$J_\Pi = \Pi_{A_iB_i \to A_oB_o}\left(\Phi_{A_iB_i:A_i'B_i'}\right)$$

- Unassisted code (UA):

$$\Pi_{A_iB_i \to A_oB_o} = \mathcal{E}_{A_i \to A_o} \otimes \mathcal{D}_{B_i \to B_o}.$$

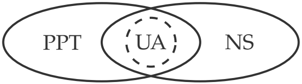
- Positive partial transpose preserving (PPT) code: [Rains, 1999; Rains, 2001]

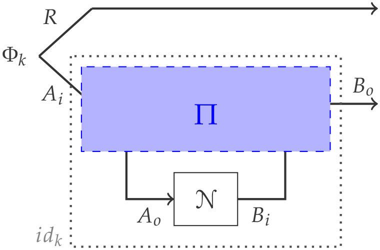
$$\Pi_{A_iB_i \to A_oB_o} \text{PPT operation} \quad J_\Pi^{T_{B_iB_o}} \geq 0.$$

- Non-signalling (NS) code: [Leung, Matthews, 2015; Duan, Winter, 2016]

$$\operatorname{Tr}_{A_o} J_\Pi = \frac{\mathbb{1}_{A_i}}{d_{A_i}} \otimes \operatorname{Tr}_{A_iA_o} J_\Pi, \quad (A \not\to B)$$

$$\operatorname{Tr}_{B_o} J_\Pi = \frac{\mathbb{1}_{B_i}}{d_{B_i}} \otimes \operatorname{Tr}_{B_iB_o} J_\Pi, \quad (B \not\to A)$$





Maximum channel fidelity

$$F_\Omega(\mathcal{N}, k) := \sup_{\Pi \in \Omega} \operatorname{Tr}\big(\underbrace{\Phi_k}_{\text{input}} \cdot \underbrace{\Pi \circ \mathcal{N}(\Phi_k)}_{\text{output}}\big).$$

One-shot quantum capacity

error tolerance

$$Q^{(1)}_\Omega(\mathcal{N}, \varepsilon) := \log \max \{k : F_\Omega(\mathcal{N}, k) \geq 1 - \varepsilon\}.$$

(Asymptotic) quantum capacity

$$Q_\Omega(\mathcal{N}) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} Q^{(1)}_\Omega(\mathcal{N}^{\otimes n}, \varepsilon).$$

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[Leung, Matthews, 2015]

$$F_\Omega(\mathcal{N},k) = \max \operatorname{Tr} J_\mathcal{N} W_{AB} \text{ s.t. } 0 \le W_{AB} \le \rho_A \otimes \mathbb{1}_B, \operatorname{Tr}\rho_A = 1,$$

$$\textbf{PPT:} -k^{-1}\rho_A \otimes \mathbb{1}_B \le W_{AB}^{T_B} \le k^{-1}\rho_A \otimes \mathbb{1}_B,\ \textbf{NS:} \operatorname{Tr}_A W_{AB} = k^{-2}\mathbb{1}_B.$$

Optimization characterization

$$Q^{(1)}_{PPT}(\mathcal{N},\varepsilon) = -\log \min m$$

$$\text{s.t. } \operatorname{Tr} J_\mathcal{N} W_{AB} \ge 1-\varepsilon, 0 \le W_{AB} \le \rho_A \otimes \mathbb{1}_B,$$

$$\operatorname{Tr}\rho_A = 1, -m\rho_A \otimes \mathbb{1}_B \le W_{AB}^{T_B} \le m\rho_A \otimes \mathbb{1}_B,$$

$$\left[\operatorname{Tr}_A W_{AB} = m^2 \mathbb{1}_B \quad \text{NS condition}\right]$$

Non-linear terms

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$$
\begin{aligned}
Q^{(1)}_{PPT}(\mathcal{N},\varepsilon) = -\log \min\ & m \\
\text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\
& \operatorname{Tr} \rho_A = 1, -m\rho_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B} \leq m\rho_A \otimes \mathbb{1}_B. \\
& \left[\operatorname{Tr}_A W_{AB} = m^2 \mathbb{1}_B.\ \text{NS condition}\right]
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
g(\mathcal{N},\varepsilon) := \min \operatorname{Tr} S_A \\
\text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\
& \operatorname{Tr} \rho_A = 1, -S_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B} \leq S_A \otimes \mathbb{1}_B.
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\widetilde{g}(\mathcal{N},\varepsilon) := \min \operatorname{Tr} S_A \\
\text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\
& \operatorname{Tr} \rho_A = 1, -S_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B} \leq S_A \otimes \mathbb{1}_B, \\
& \operatorname{Tr}_A W_{AB} = t\mathbb{1}_B.
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\widehat{g}(\mathcal{N},\varepsilon) := \min \operatorname{Tr} S_A \\
\text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\
& \operatorname{Tr} \rho_A = 1, -S_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B} \leq S_A \otimes \mathbb{1}_B, \\
& \operatorname{Tr}_A W_{AB} = t\mathbb{1}_B, t \geq \widehat{m}^2, \\
& \left(Q^{(1)}_{PPT\cap NS}(\mathcal{N},\varepsilon) \leq -\log \widehat{m}\right).
\end{aligned} \tag{4}
$$

Main result 1: SDP converse bounds for one-shot quantum capacity

[Tomamichel, Berta, Renes, 2016]

$$
\begin{aligned}
f(\mathcal{N},\varepsilon) = \min \operatorname{Tr} S_A & \\
\text{s.t. } \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, S_A, \Theta_{AB} \geq 0, \operatorname{Tr} \rho_A = 1, & \\
0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, S_A \otimes \mathbb{1}_B \geq W_{AB} + \Theta_{AB}^{T_B}. &
\end{aligned}
\tag{5}
$$

Theorem

For any quantum channel $\mathcal{N}$ and error tolerance ε, the inequality chain holds

$$
\begin{aligned}
Q^{(1)}(\mathcal{N},\varepsilon) \leq Q^{(1)}_{PPT\cap NS}(\mathcal{N},\varepsilon) & \\
\leq -\log \widehat{g}(\mathcal{N},\varepsilon) \leq -\log \widetilde{g}(\mathcal{N},\varepsilon) \leq -\log g(\mathcal{N},\varepsilon) \leq -\log f(\mathcal{N},\varepsilon). &
\end{aligned}
\tag{6}
$$

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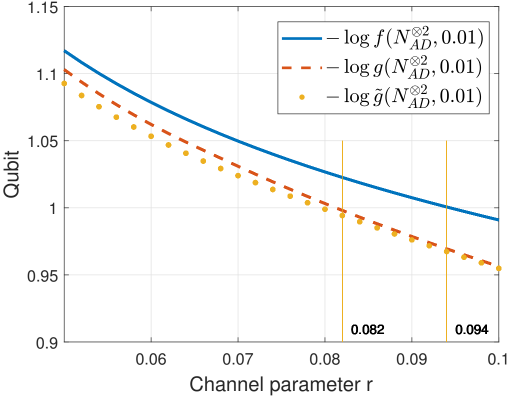
Amplitude damping channel $\mathcal{N}_{AD} = \sum_{i=0}^{1} E_i \cdot E_i^\dagger$ with

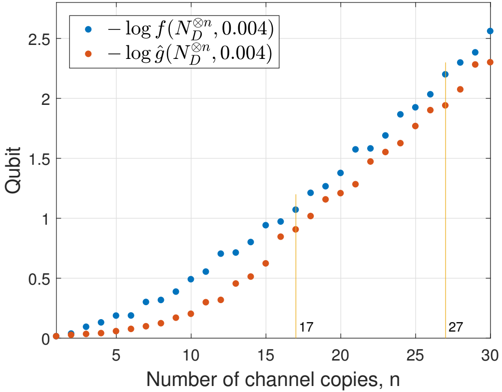
$$E_0 = |0\rangle\langle 0| + \sqrt{1-r}|1\rangle\langle 1| \quad E_1 = \sqrt{r}|0\rangle\langle 1|, \quad 0 \le r \le 1$$



Example: Qubit depolarizing channel

Qubit depolarizing channel $\mathcal{N}_D(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$,

where X, Y, Z are Pauli matrices.



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Asymptotic quantum capacity

SDP strong converse bound for quantum capacity

$$Q_{PPT}^{(1)}(\mathcal{N},\varepsilon) = -\log \min m$$
$$\text{s.t. } \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B,$$
$$\operatorname{Tr}\rho_A = 1, -m\rho_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B} \leq m\rho_A \otimes \mathbb{1}_B.$$

Take $R_{AB} = W_{AB}/m$ and throw away the condition $W_{AB} \leq \rho_A \otimes \mathbb{1}_B$, we obtain an additive SDP upper bound $Q_{PPT}^{(1)}(\mathcal{N},\varepsilon) \leq Q_\Gamma(\mathcal{N}) - \log(1-\varepsilon)$, where

$$Q_\Gamma(\mathcal{N}) = \log \max \operatorname{Tr} J_{\mathcal{N}} R_{AB}$$
$$\text{s.t. } R_{AB}, \rho_A \geq 0, \operatorname{Tr}\rho_A = 1, \tag{7}$$
$$-\rho_A \otimes \mathbb{1}_B \leq R_{AB}^{T_B} \leq \rho_A \otimes \mathbb{1}_B.$$

- Additivity: $Q_\Gamma(\mathcal{N}\otimes\mathcal{M}) = Q_\Gamma(\mathcal{N}) + Q_\Gamma(\mathcal{M})$ (by utilizing SDP duality).
- Converse bound for $Q(\mathcal{N})$: $Q(\mathcal{N}) \leq Q_{PPT}(\mathcal{N}) \leq Q_\Gamma(\mathcal{N})$.
- For noiseless quantum channel $\mathcal{I}_d$, $Q(\mathcal{I}_d) = Q_\Gamma(\mathcal{I}_d) = \log_2 d$.
- Strong converse: denote the n-shot optimal rate as r, then (r, n, ε) satisfies $nr \leq nQ_\Gamma(\mathcal{N}) - \log(1-\varepsilon)$, which implies $\varepsilon \geq 1 - 2^{n(Q_\Gamma(\mathcal{N})-r)}$.

Main result 2: SDP strong converse bound for quantum capacity

Theorem (SDP strong converse bound for Q)

For any quantum channel $\mathcal{N}$,

$$Q(\mathcal{N}) \leq Q_\Gamma(\mathcal{N}) = \log \max \operatorname{Tr} J_{\mathcal{N}} R_{AB}$$
$$\text{s.t. } R_{AB}, \rho_A \geq 0, \operatorname{Tr} \rho_A = 1,$$
$$-\rho_A \otimes \mathbb{1}_B \leq R_{AB}^{T_B} \leq \rho_A \otimes \mathbb{1}_B.$$

The fidelity of transmission goes to zero if the rate exceeds $Q_\Gamma(\mathcal{N})$.

How to understand $Q_\Gamma(\mathcal{N})$?

Entanglement measure

$$Q_\Gamma(\mathcal{N}) = \max_{\rho_A \in \mathcal{S}(A)} E_W\left(\mathcal{N}_{A'\to B}\left(\phi_{AA'}\right)\right)$$
$$= \max_{\rho \in \mathcal{S}(A)} \min_{\sigma \in \mathrm{PPT}'} D_{\max}\left(\mathcal{N}_{A'\to B}\left(\phi_{AA'}\right) \| \sigma\right)$$

where $E_W(\rho) := \log \max \left\{\operatorname{Tr} \rho R_{AB} : -\mathbb{1}_{AB} \leq R_{AB}^{T_B} \leq \mathbb{1}_{AB}, R_{AB} \geq 0\right\}$, [Wang, Duan, 2016], $\phi_{AA'}$ is a purification of ρ_A and $\mathrm{PPT}' = \left\{\sigma \geq 0 : \left\|\sigma^{T_B}\right\|_1 \leq 1\right\}$.

Remark: For any EB channel $\mathcal{N}$, $Q_\Gamma(\mathcal{N}) = 0$. If $Q_E(\mathcal{N}) \neq 0$, $Q_\Gamma(\mathcal{N}) < Q_E(\mathcal{N})$.

Comparison with other bounds

Rains information [Tomamichel, Wilde, Winter, 2016]

$$R(\mathcal{N}) := \max_{\rho \in \mathcal{S}(A)} \min_{\sigma \in \mathrm{PPT}'} D\left(\mathcal{N}_{A' \to B}(\phi_{AA'}) \| \sigma\right)$$

$$Q_\Gamma(\mathcal{N}) = \max_{\rho \in \mathcal{S}(A)} \min_{\sigma \in \mathrm{PPT}'} D_{\max}\left(\mathcal{N}_{A' \to B}(\phi_{AA'}) \| \sigma\right)$$

Due to the fact that $D(\rho\|\sigma) \leq D_{\max}(\rho\|\sigma)$ [Datta, 2009], we have $\underline{R(\mathcal{N}) \leq Q_\Gamma(\mathcal{N})}$.

- $R(\mathcal{N})$ strong converse but not known to be efficiently computable in general.
- $Q_\Gamma(\mathcal{N})$ strong converse and **efficiently computable** in general.

Comparison with other bounds

◎ Partial Transposition bound [Holevo, Werner, 2001]

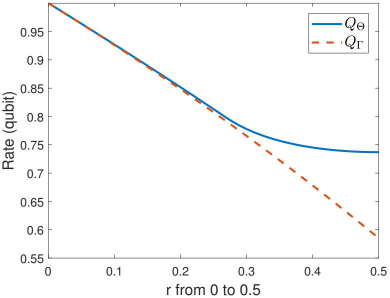
$$Q(\mathcal{N}) \le Q_\Theta(\mathcal{N}) = \log \|\mathcal{N} \circ T\|_\diamond ,$$

where T is the transpose map, $\|\mathcal{N}\|_\diamond = \|\mathcal{N} \otimes id\|_1$ and can be characterized by SDP from [Watrous, 2012].

Improved efficiently computable bound

For any quantum channel $\mathcal{N}$, it holds $Q_\Gamma(\mathcal{N}) \le Q_\Theta(\mathcal{N})$.

Example: $\mathcal{N}_r = \sum_i E_i \cdot E_i^\dagger$ where $E_0 = |0\rangle\langle 0| + \sqrt{r}|1\rangle\langle 1|,\ E_1 = \sqrt{1-r}|0\rangle\langle 1| + |1\rangle\langle 2|$.



Converse bounds comparison

For any quantum channel $\mathcal{N}$, it holds

$$Q(\mathcal{N}) \le R(\mathcal{N}) \le Q_\Gamma(\mathcal{N}) \le Q_\Theta(\mathcal{N}) .$$

Known converse bounds

	Strong converse	Efficiently computable	For general channels
Q_Γ	✓	✓	✓
R	✓	? (max-min)	✓
ε-DEG	?	✓	✗
E_C	✓	? (regularization)	✓
Q_E	✓	✓	✓
Q_{ss}	?	? (unbounded dimension)	✓
Q_Θ	✓	✓	✓

- Q_Γ: SDP strong converse bound in this talk.
- R: Rains information [Tomamichel, Wilde, Winter, 2017]
- ε-DEG: Epsilon degradable bound [Sutter, Scholz, Winter, Renner, 2014]
- E_C: Channel's entanglement cost [Berta, Brandão, Christandl,Wehner, 2013]
- Q_E: Entanglement assisted quantum capacity [Bennett, Devetak, Harrow, Shor, Winter,2014; Berta, Christandl, Renner,2011]
- Q_{ss}: Quantum capacity with symmetric side channels [Smith, Smolin, Winter, 2008]
- Q_Θ: Partial transposition bound [Holevo,Werner, 2001]
- $\exists\ \mathcal{N}, Q_\Gamma(\mathcal{N}) < \varepsilon\text{-DEG}(\mathcal{N})$.

Summary of results

Theorem (SDP converse bounds for finite blocklength Q)

For any quantum channel $\mathcal{N}$ and error tolerance ε, the inequality chain holds

$$
\begin{aligned}
Q^{(1)}(\mathcal{N},\varepsilon) &\leq Q^{(1)}_{PPT\cap NS}(\mathcal{N},\varepsilon) \\
&\leq -\log \widehat{g}(\mathcal{N},\varepsilon) \leq -\log \widetilde{g}(\mathcal{N},\varepsilon) \leq -\log g(\mathcal{N},\varepsilon) \leq -\log f(\mathcal{N},\varepsilon).
\end{aligned}
$$

Theorem (SDP strong converse bound for Q)

For any quantum channel $\mathcal{N}$,

$$
\begin{aligned}
Q(\mathcal{N}) \leq Q_\Gamma(\mathcal{N}) = \log \max \operatorname{Tr} J_\mathcal{N} R_{AB} \\
\text{s.t. } R_{AB}, \rho_A \geq 0, \operatorname{Tr} \rho_A = 1, \\
-\rho_A \otimes \mathbb{1}_B \leq R_{AB}^{T_B} \leq \rho_A \otimes \mathbb{1}_B.
\end{aligned}
$$

$$
Q(\mathcal{N}) \leq R(\mathcal{N}) \leq Q_\Gamma(\mathcal{N}) \leq Q_\Theta(\mathcal{N}).
$$

Open questions and future works

- How to apply our relaxation technique to Gaussian channels?
- Q_Γ does not work well for depolarizing channels. Can we obtain a better result from the linear programs $\widehat{g}$, $\widetilde{g}$ or g?

THE END

THANK YOU!

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