Distillation of quantum coherence

(1711.10512 & 1804.09500)

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RMS:QI workshop 2018, JILA

Based on joint works with
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- Coherence theory background
- Deterministic setting
- Probabilistic setting
- Summary and discussions

Resource theory:

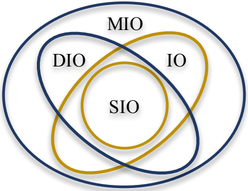
- Free states, e.g. separable states;
- Resource states, e.g. entangled states like $|\Phi_m\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |ii\rangle$;
- Free operations, e.g. LOCC, SEP, SEPP, PPT...

A special case of resource theory:

- Free states: incoherent states $\mathcal{I} := \{\rho \geq 0 : \operatorname{Tr} \rho = 1, \rho = \Delta(\rho)\}$;
- Resource states: coherent state like $|\Psi_m\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i\rangle$.
- Free operatioins, e.g. SIO, IO, DIO, MIO.

Quantum coherence as a resource:

- Implement the Deutsch-Jozsa algorithm [Hillery, 2016];
- Quantum state merging [Streltsov et al., 2016];
- Quantum channel simulation [Díaz et al., 2018];
- ...



Semidefinite conditions for MIO and DIO:

- MIO: $\mathscr{E}(|i\rangle\langle i|) = \Delta(\mathscr{E}(|i\rangle\langle i|))$ for all i.
- DIO: MIO and $\Delta\left(\mathscr{E}\left(|i\rangle\langle j|\right)\right) = 0$ for $i \neq j$.

- Maximally incoherent operations (MIO): $\mathscr{E}(\mathcal{I}) \subseteq \mathcal{I}$;
- Dephasing-covariant incoherent operations (DIO): $[\mathscr{E}, \Delta] = 0$;
- Incoherent operations (IO):

  Kraus operators $\{E_i\}$ such that $\frac{E_i \rho E_i^\dagger}{\operatorname{Tr} E_i \rho E_i^\dagger} \in \mathcal{I}$ for all $\rho \in \mathcal{I}$;
- Strictly incoherent operations (SIO): both E_i and $E_i^\dagger$ are incoherent.

More about quantum coherence theory refer to [Streltsov, Adesso, Plenio, 2017] and quantum resource theory refer to [Chitambar and Gour, 2018]...

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Deterministic setting

The fidelity of coherence distillation under the class of operations Ω is defined by

$$F_{\Omega}(\rho, m) := \max_{\Pi \in \Omega} \operatorname{Tr} \Pi(\rho) \Psi_m. \tag{1}$$

The one-shot ε-error distillable coherence under the class of operation Ω is defined as

$$C_{d,\Omega}^{(1),\varepsilon}(\rho) := \log \max \left\{ m \in \mathbb{N} \mid F_{\Omega}(\rho, m) \geq 1 - \varepsilon \right\}. \tag{2}$$

The asymptotic distillable coherence can be given as

$$C_{d,\Omega}(\rho) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} C_{d,\Omega}^{(1),\varepsilon}\left(\rho^{\otimes n}\right). \tag{3}$$

Similarly we can define the coherence cost of a quantum state $C_{c,\Omega}(\rho)$.

$$C_{d,\mathrm{DIO}}(\rho) \leq C_{d,\mathrm{MIO}}(\rho) \leq C_{c,\mathrm{MIO}}(\rho) \leq C_{c,\mathrm{DIO}}(\rho)$$

$$C_r(\rho) := \min_{\sigma \in \mathcal{I}} D(\rho \| \sigma) = D(\rho \| \Delta(\rho))$$

[Winter and Yang, 2016] $\rightarrow C_r(\rho)$

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[Winter and Yang, 2016] → $C_r(\rho)$ ← [Zhao et al., 2017]

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[Chitambar, 2017]

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[Chitambar, 2017]

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Reversibility for entanglement theory [Brandão and Plenio, 2010] and other resource theory [Brandão and Gour, 2015] only known under **resource (asymptotically) non-generating maps**. The case of coherence theory set a difference from the others.

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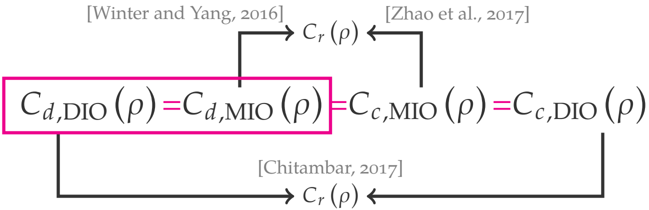

Reversibility for entanglement theory [Brandão and Plenio, 2010] and other resource theory [Brandão and Gour, 2015] only known under **resource (asymptotically) non-generating maps**. The case of coherence theory set a difference from the others.

Theorem

For any state ρ and operation class $\Omega \in \{\mathrm{MIO}, \mathrm{DIO}\}$, the fidelity of coherence distillation and the one-shot distillable coherence can both be written as the following SDPs:

$$F_\Omega(\rho, m) = \max\left\{\operatorname{Tr} G\rho \,\middle|\, 0 \le G \le \mathbb{1},\ \Delta(G) = \frac{1}{m}\mathbb{1}\right\}, \tag{4}$$

$$C_{d,\Omega}^{(1),\varepsilon}(\rho) = -\log\min\left\{\eta \,\middle|\, \operatorname{Tr} G\rho \ge 1-\varepsilon, 0 \le G \le \mathbb{1}, \Delta(G) = \eta\mathbb{1}\right\}. \tag{5}$$

Proof ingredients: symmetry of Ψ_m and semidefinite conditions for MIO.

Then we observe that the optimal operation MIO admits the structure of DIO.

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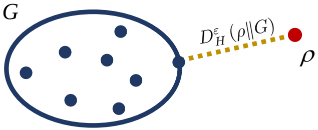

Denote the set of diagonal Hermitian operators with unit trace,

$$\mathcal{J} = \{G \mid \operatorname{Tr} G = 1,\ \Delta(G) = G\}.$$

Then $$C_{d,\Omega}^{(1),\varepsilon}(\rho) = \min_{G\in\mathcal{J}} D_H^\varepsilon(\rho\|G).$$

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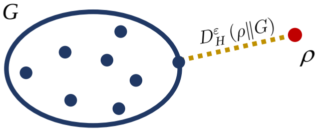

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Then $\quad C_{d,\Omega}^{(1),\varepsilon}(\rho) = \min_{G\in\mathcal{J}} D_H^\varepsilon(\rho\|G).$

Remark: Similar characterizations independently found by Winter's group.

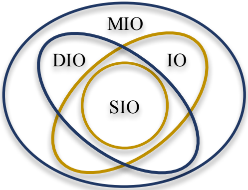
For the case of pure states, we go beyond MIO and DIO.

Theorem

For any pure state $|\psi\rangle$, we have

$$F_{\text{SIO}}(\psi, m) = F_{\text{IO}}(\psi, m) = F_{\text{DIO}}(\psi, m) = F_{\text{MIO}}(\psi, m),$$

$$C^{(1),\varepsilon}_{d,\text{SIO}}(\psi) = C^{(1),\varepsilon}_{d,\text{IO}}(\psi) = C^{(1),\varepsilon}_{d,\text{DIO}}(\psi) = C^{(1),\varepsilon}_{d,\text{MIO}}(\psi).$$



Sketch of proof: $F_{\text{SIO}}(\psi, m) = F_{\text{MIO}}(\psi, m)$

- Introduce a intermediate quantity $\frac{1}{m}\||\psi\rangle\|^2_{[m]}$ which admits $\max\{\operatorname{Tr}\psi W : 0 \le W \le \mathbb{1}, \Delta(W) \le \frac{1}{m}\mathbb{1}\}$;
- Compare SDPs and have $F_{\text{MIO}}(\psi, m) \le \frac{1}{m}\||\psi\rangle\|^2_{[m]}$;
- Construct $|\eta\rangle$ such that $\lambda_\psi \prec \lambda_\eta$ ($\psi \xrightarrow{\text{SIO}} \eta$) and $F(\eta, \Psi_m) = \frac{1}{m}\||\psi\rangle\|^2_{[m]}$, thus $F_{\text{SIO}}(\psi, m) \ge \frac{1}{m}\||\psi\rangle\|^2_{[m]}$.

More details refer to arXiv: 1711.10512.

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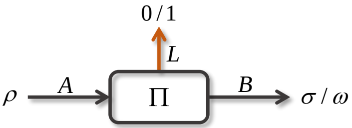
Probabilistic setting



Resource state: ρ

Target state: Ψ_m

Garbage state: ω

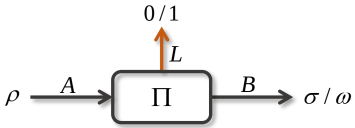
Flag register: L

For any triple (ρ, m, ε), the **maximum success probability** of coherence distillation under the operation class $\Omega \in \{\text{SIO}, \text{IO}, \text{DIO}, \text{MIO}\}$ is defined as

$$P_\Omega(\rho \to \Psi_m, \varepsilon) := \max \; p \tag{6a}$$

$$\text{s.t.} \; \Pi_{A\to LB}(\rho) = p|0\rangle\langle 0|_L \otimes \sigma + (1-p)\,|1\rangle\langle 1|_L \otimes \omega, \tag{6b}$$

$$F(\sigma, \Psi_m) \geq 1-\varepsilon, \; \Pi \in \Omega, \; 0 \leq p \leq 1. \tag{6c}$$



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Twirling $\mathscr{T}(\rho) = \frac{1}{d!}\sum_i P_i \rho P_i$ where P_i is permutation of reference basis.

Simplification without compromising the maximum success probability:

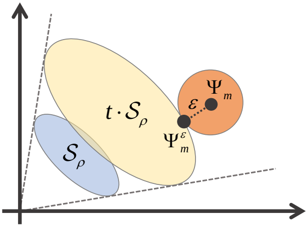
- Garbage state $\omega \xrightarrow{\Delta} \Delta(\rho) \xrightarrow{\mathscr{T}} \mathbb{1}/m$;
- Optimal output state $\sigma \xrightarrow{\mathscr{T}} \Psi_m^\varepsilon$ where $\Psi_m^\varepsilon := (1-\varepsilon)\Psi_m + \varepsilon(\mathbb{1} - \Psi_m)/(m-1)$;
- $P_\Omega(\rho \to \Psi_m, \varepsilon) = P_\Omega(\rho \to \Psi_m^\varepsilon, 0)$.

Theorem

For any triplet (ρ, m, ε) and operation class Ω, the maximal success probability is given by

$$P_\Omega(\rho \rightarrow \Psi_m, \varepsilon)^{-1} = \min\{t \in \mathbb{R}_+ \mid \Psi_m^\varepsilon \in t \cdot S_\rho\} \quad \textit{where} \tag{7}$$

$S_\rho := \{\mathscr{E}(\rho) \mid \mathscr{E} \in \Omega_{\text{sub}}\}$ *is the set of all output operators of ρ under the operation class Ω_{sub} (completely positive and trace-nonincreasing maps (sub-operations)).*



Intuition: the closer the state ρ to Ψ_m (more coherent) $\Rightarrow$ the less we need to expand the set S_ρ $\Rightarrow$ the larger success probability we can obtain.

Theorem

For any triplet (ρ, m, ε), the maximal success probability of distillation under MIO/DIO are

$$
\begin{aligned}
P_{\mathrm{MIO}}(\rho \to \Psi_m, \varepsilon) = \max.\ & \operatorname{Tr} G\rho \\
\text{s.t.}\ & \Delta(G) = m\Delta(C), && \text{(8a)} \\
& 0 \le C \le G \le \mathbb{1}, && \text{(8b)} \\
& \operatorname{Tr} C\rho \ge (1-\varepsilon)\operatorname{Tr} G\rho. && \text{(8c)} \\
P_{\mathrm{DIO}}(\rho \to \Psi_m, \varepsilon) = \max.\ & \operatorname{Tr} G\rho \\
\text{s.t.}\ & \text{Eqs. } (8a, 8b, 8c), \\
& G = \Delta(G).
\end{aligned}
$$

Proof ingredients: symmetry of Ψ_m^ε and semidefinite conditions for MIO and DIO.

Zero-error case: $P_{\mathrm{MIO}}\left(\rho \to \Psi_m, 0\right)$

$$P_{\mathrm{MIO}}\left(\rho \to \Psi_m, \varepsilon\right) = \max.\ \operatorname{Tr} G\rho$$
$$\text{s.t.}\ \ \Delta(G) = m\Delta(C),\ 0 \leq C \leq G \leq \mathbb{1},\ \operatorname{Tr} C\rho \geq (1-\varepsilon) \operatorname{Tr} G\rho.$$

Theorem

*For any triplet $\left(\rho, m, 0\right)$ with a **full-rank** state ρ, it holds that $P_{\mathrm{MIO}}\left(\rho \to \Psi_m, 0\right) = 0$.*

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- Any generic density matrix has full rank;
- Non-continuity: $|P_{\mathrm{MIO}}(\Psi_m^{\varepsilon} \to \Psi_m, 0) - P_{\mathrm{MIO}}(\Psi_m \to \Psi_m, 0)| = 1$;
- Depolarizing noise: $\alpha \cdot \rho + (1-\alpha)\mathbb{1}/m$ is full rank;

Zero-error case: $P_{\mathrm{MIO}}(\rho \to \Psi_m, 0)$

$$P_{\mathrm{MIO}}(\rho \to \Psi_m, \varepsilon) = \max.\ \operatorname{Tr} G\rho$$
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Theorem

For any triplet $(\varphi, m, 0)$ with a coherent ***pure state*** $|\varphi\rangle = \sum_{i=1}^n \varphi_i |i\rangle$, $\varphi_i \neq 0$, $n \geq 2$, *it holds*

$$P_{\mathrm{MIO}}(\varphi \to \Psi_m, 0) \geq \frac{n^2}{\sum_{i=1}^n |\varphi_i|^{-2}} \left\| \frac{n-m}{n-1}\widetilde{\varphi} + \frac{n(m-1)}{n-1}\Delta(\widetilde{\varphi}) \right\|_\infty^{-1} \geq \frac{n^2}{m\left(\sum_{i=1}^n |\varphi_i|^{-2}\right)} > 0,$$

where $|\widetilde{\varphi}\rangle := \frac{1}{\sqrt{s}} \sum_{i=1}^n \frac{\varphi_i}{|\varphi_i|^2} |i\rangle$ *with* $s = \sum_{j=1}^n |\varphi_j|^{-2}$.

Zero-error case: $P_{\mathrm{MIO}}\left(\rho \to \Psi_m, 0\right)$

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- $P_{\mathrm{MIO}}\left(\Psi_2 \to \Psi_{10^6}, 0\right) \geq \frac{1}{10^6 - 1}$.

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- $P_{\mathrm{MIO}}\left(\Psi_2 \to \Psi_{10^6}, 0\right) \ge \frac{1}{10^6 - 1}$. **Gambling!** 

Fundamental difference between MIO and DIO, **contrast to the deterministic case**:

- $P_{\mathrm{MIO}}(\Psi_n \to \Psi_{n+1}, 0) \ge \frac{n-1}{n} \to 1;$
- $P_{\mathrm{DIO}}(\Psi_n \to \Psi_{n+1}, 0) = 0.$

Zero-error case: $P_{\mathrm{DIO}}(\rho \to \Psi_m, 0)$

Recall some results in entanglement theory:

- $|\varphi\rangle = \sum_{i=1}^{n} \sqrt{\varphi_i}|ii\rangle$, φ_i nonincreasing, $\lambda_\varphi := (\varphi_i)_i$;
  $|\psi\rangle = \sum_{i=1}^{n} \sqrt{\psi_i}|ii\rangle$, ψ_i nonincreasing, $\lambda_\psi := (\psi_i)_i$;
- [Nielsen, 1999] $\varphi \xrightarrow{LOCC} \psi$ iff $\lambda_\varphi \prec \lambda_\psi$;
- [Vidal, 1999] $P_{\mathrm{LOCC}}(\varphi \to \psi, 0) = \min_{k\in[1,n]} \frac{\sum_{i=k}^{n} \varphi_i}{\sum_{i=k}^{n} \psi_i}$.

For any pure state $|\varphi\rangle = \sum_{i=1}^{n} \sqrt{\varphi_i}|i\rangle$, it holds [Chitambar and Gour, 2016; Zhu et al, 2017]

$$P_{\mathrm{(S)IO}}(\varphi \to \Psi_m, 0) = \begin{cases} 0 & \text{if } \operatorname{rank}\Delta(\varphi) < m, \\ \min\limits_{k\in[1,m]} \dfrac{m}{k} \sum\limits_{i=m-k+1}^{d} \varphi_i & \text{otherwise.} \end{cases} \tag{9}$$

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Theorem

For any pure state φ and any m, we have

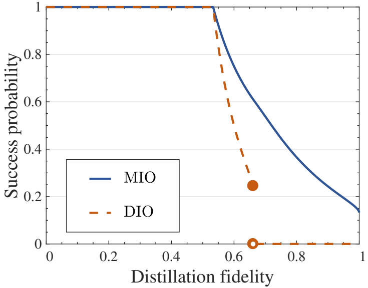
$$P_{\text{DIO}}(\varphi \to \Psi_m, 0) = P_{\text{(S)IO}}(\varphi \to \Psi_m, 0). \tag{10}$$

Sketch of proof: to show $P_{\text{DIO}}(\varphi \to \Psi_m, 0) \le P_{\text{(S)IO}}(\varphi \to \Psi_m, 0)$, use the minimization problem for DIO and construct feasible solutions.

Theorem

For any pure state $|\varphi\rangle = \sum_{i=1}^{n} \varphi_i |i\rangle$ *with nonzero coefficients* φ_i*, it holds that*

$$P_{\mathrm{DIO}}(\varphi \to \Psi_m, \varepsilon) \begin{cases} > 0 & \text{if } n \geq m \text{ or if } n < m \text{ and } \varepsilon \geq 1 - \frac{n}{m}, \\ = 0 & \text{if } n < m \text{ and } \varepsilon < 1 - \frac{n}{m}. \end{cases}$$



$$(|0\rangle + 3|1\rangle)/\sqrt{10} \to \Psi_3$$

Non-tradeoff phenomenon

Theorem

For any pure state $|\varphi\rangle = \sum_{i=1}^{n} \varphi_i |i\rangle$ *with nonzero coefficients* φ_i*, it holds that*

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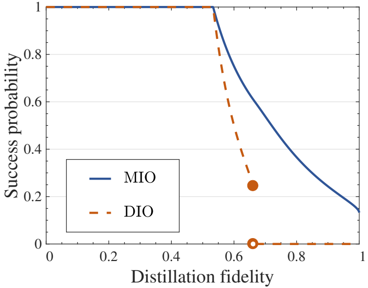


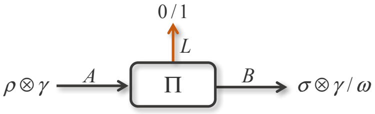
$$\left(|0\rangle + 3|1\rangle\right)/\sqrt{10} \to \Psi_3$$

This is "analogous" to the (pretty) strong converse theorem in channel coding theory: the coding success probability goes to zero if the coding rate exceeds the capacity of the channel.

Distillation with catalytic assistance

$\rho \not\longrightarrow \sigma$ but $\rho \otimes \gamma \longrightarrow \sigma \otimes \gamma$

$P_\Omega\left(\rho \otimes \gamma \to \Psi_m \otimes \gamma, 0\right) > P_\Omega\left(\rho \to \Psi_m, 0\right)$



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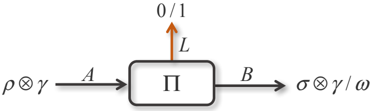

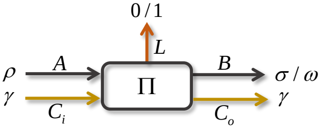

$P_\Omega(\rho \xrightarrow{\gamma} \Psi_m, \varepsilon) > P_\Omega(\rho \to \Psi_m, \varepsilon)$

Distillation with catalytic assistance

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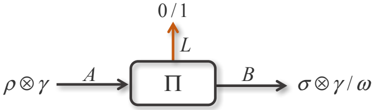
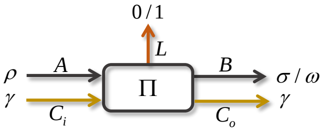
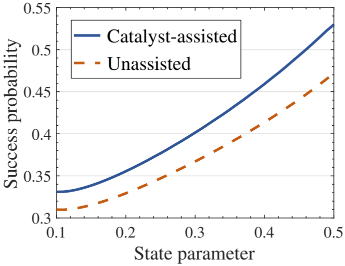




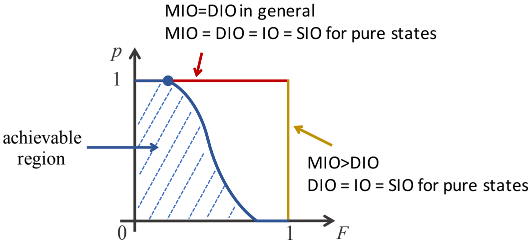
$P_\Omega(\rho \xrightarrow{\gamma} \Psi_m, \varepsilon) > P_\Omega(\rho \to \Psi_m, \varepsilon)$



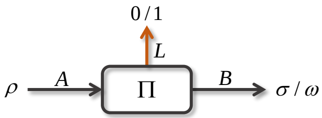
Taking as an example the two-qubit state $\rho = q \cdot v_1 + (1 - q)\, v_2$ and $\gamma = \Psi_2$ with

$$|v_1\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)$$

$$|v_2\rangle = \frac{1}{5\sqrt{2}}\left(2|00\rangle + 6|01\rangle - 3|10\rangle + |11\rangle\right)$$



- SDP characterizations for one-shot distillation rate and maximum success probability under MIO and DIO;
- For $\Omega \in \{\text{DIO}, \text{MIO}\}$, $C_{d,\Omega}^{(1),\varepsilon}(\rho) = \min_{G \in \mathcal{J}} D_H^{\varepsilon}(\rho \| G)$;
- No-go theorem: no full-rank state can be perfectly transformed into Ψ_m under free operations, not even probabilistically!
- There is a non-tradeoff phenomenon between fidelity and success probability under DIO.



- Can we recycle the garbage state ω if the distillation process fails?
- Any interesting phenomenon for probabilistic coherence dilution?
- More detailed analysis of catalytic scenario?

Thanks for your attention!

See arXiv:

1711.10512 & 1804.09500

for more details